HORNSBY GIRLS HIGH SCHOOL

Mathematics Extension 2

Year 12 Higher School Certificate
Trial Examination Term 3 2022

STUDENT NUMBER: ______________________________

General Instructions

- Reading Time – 10 minutes
- Working Time – 3 hours
- Write using black pen
  Black pen is preferred
- NESA-approved calculators and drawing templates may be used
- A reference sheet is provided separately
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations
- Marks may be deducted for untidy and poorly arranged work
- Do not use correction fluid or tape
- Do not remove this paper from the examination room

Total marks – 102

Section I Pages 3 – 6

10 marks

Attempt Questions 1 – 10

Answer on the Objective Response Answer Sheet provided

Section II Pages 7 – 12

90 marks

Attempt Questions 11 – 16

Start each question in a new writing booklet

Write your student number on every writing booklet

Question Total	***1-10***	***11***	***12***	***13***	***14***	***15***	***16***	***Total***
	/10	*/15*	*/15*	*/17*	*/15*	*/15*	*/15*	*/102*

This assessment task constitutes 30% of the Higher School Certificate Course School Assessment

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the Objective Response answer sheet for Questions 1 – 10

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1 The angles made between $\underset{\sim}{v} = \overrightarrow{OV}$ where V is the point (5, -3, 3) with the positive direction of x, y and z axes respectively are

(A) $40°19'$, $117°14'$ and $62°46'$

(B) $40°19'$, $62°14'$ and $62°46'$

(C) $180°$, $53°8'$ and $53°8'$

(D) $0°$, $126°52'$ and $53°8'$.

2 By using the mod-arg form or otherwise, $\dfrac{\left(1-\sqrt{3}i\right)^6}{(1+i)^4}$ can be expressed as:

(A) $16+0i$

(B) $-16+0i$

(C) $4+0i$

(D) $-4+0i$

3. It is stated "If anyone has tested positive for COVID 19, then they are to isolate for 7 days". The converse of that statement is:

(A) If anyone has not tested positive for COVID 19, then they are not to isolate for 7 days.

(B) If anyone has not isolated for 7 days, then they are not tested positive for COVID 19.

(C) If anyone has isolated for 7 days, then they are tested positive for COVID 19.

(D) If anyone has isolated for 7 days, if and only if they are tested positive for COVID 19.

4 The integral of $\int \frac{4x^3}{1+x^8}\ dx =$

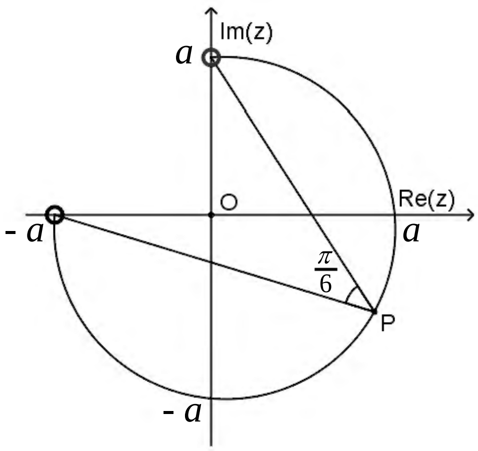
(A) $4\ln\left(1+x^8\right)+C$

(B) $\frac{1}{2}\ln\left(1+x^8\right)+C$

(C) $4\tan^{-1}\left(x^4\right)+C$

(D) $\tan^{-1}\left(x^4\right)+C$

5 Which of the equations best represents the given diagram of complex number z, where $a \in Z$?



NOT TO SCALE

(A) $\arg\left(\frac{z-a}{z+ai}\right)=\frac{\pi}{6}$

(B) $\arg\left(\frac{z+ai}{z-a}\right)=\frac{\pi}{6}$

(C) $\arg\left(\frac{z+ai}{z-a}\right)=-\frac{\pi}{6}$

(D) $\arg\left(\frac{z+a}{z-ai}\right)=\frac{\pi}{6}$

6 The integral of $\int \ln x \; dx$ is

(A) $x\ln x - 1 + C$

(B) $x\ln x + 1 + C$

(C) $x\ln x - x + C$

(D) $x\ln x + x + C$

7 The Cartesian equation of a sphere is $x^2 + y^2 + z^2 - 4x + 10z + 20 = 0$. The vector equation of the sphere is

(A) $\left| \underset{\sim}{r} - \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix} \right| = 3$

(C) $\left| \underset{\sim}{r} - \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} \right| = 9$

(B) $\left| \underset{\sim}{r} - \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix} \right| = 9$

(D) $\left| \underset{\sim}{r} - \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} \right| = 3$

8 Which of the following is an expression for $e^{i3\theta} + e^{i\theta}$?

(A) $2\sin\theta e^{i2\theta}$

(B) $2\cos\theta e^{i2\theta}$

(C) $2\sin 2\theta e^{i\theta}$

(D) $2\cos 2\theta e^{i\theta}$

9 Consider the statement:

$$'\exists x \in R,\ \ln x = 1 \text{ and } x > 2.'$$

Which of the following is the negation of the statement?

(A) $\exists x \in R,\ \ln x \neq 1 \text{ or } x \leq 2.$

(B) $\exists x \in R,\ \ln x \neq 1 \text{ and } x \leq 2.$

(C) $\forall x \in R,\ \ln x \neq 1 \text{ or } x \leq 2.$

(D) $\forall x \in R,\ \ln x \neq 1 \text{ and } x \leq 2.$

10 Triangle *OAB* has position vectors $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$. The triangle inequality in vector form states that $\left|\underset{\sim}{a} + \underset{\sim}{b}\right| \leq \left|\underset{\sim}{a}\right| + \left|\underset{\sim}{b}\right|$. Equality of that statement holds if and only if

(A) $\overrightarrow{OA} \parallel \overrightarrow{OB}$

(B) $\overrightarrow{OA} \perp \overrightarrow{OB}$

(C) points *O*, *A* and *B* are collinear

(D) $\overrightarrow{OA} = \overrightarrow{OB}$

End of Section I

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

Answer each question in a new writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a new writing booklet

(a) (i) Express each of the following complex numbers $z_1 = -\sqrt{2} + \sqrt{2}i$ and $z_2 = \sqrt{3} + i$ in modulus-argument form. **2**

(ii) Represent the two complex numbers z_1 and z_2 as vectors on an Argand diagram. **1**

(iii) Find the exact values of $\arg\left(\frac{z_1}{z_2}\right)$ and $\arg(z_1 + z_2)$. **2**

(b) Find $\int \sin^3 x \cos^7 x \, dx$ **2**

(c) Find $\int \frac{dx}{\sqrt{e^{2x} - 1}}$ by letting $u = \frac{1}{e^x}$ **2**

(d) (i) By applying de Moivre's theorem and by expanding $(\cos\theta + i\sin\theta)^5$, show $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ as a polynomial in $\cos\theta$. **2**

(ii) Solve the equation $\cos 5\theta = -1$ for $0 \le \theta \le 2\pi$. **4**
Hence show that $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$ and find the value of $\cos\frac{\pi}{5}\cos\frac{3\pi}{5}$.

End of the Question 11.

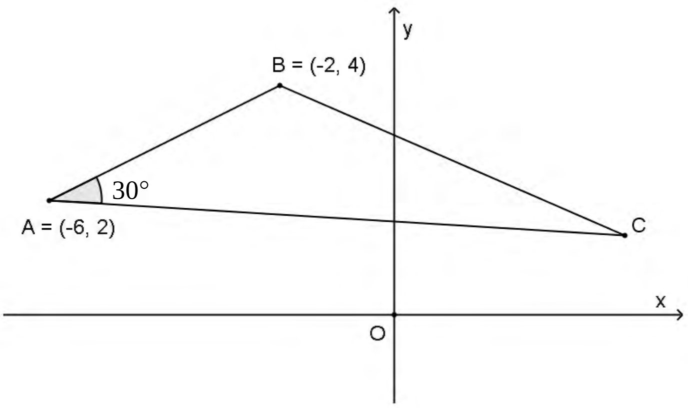
Question 12 (15 marks) Start a new writing booklet

(a) Prove by contradiction that $\log_5 7$ is irrational. **2**

(b) Find the integral $\int \sec^3 x \; dx$ **3**

(c) If ω is the complex cube root of unity, evaluate $\left(1-3\omega+\omega^2\right)\left(1+\omega-8\omega^2\right)$. **2**

(d)



NOT TO SCALE

ΔABC is drawn in the Argand diagram above where $\angle BAC = 30°$, A and B are the points (-6, 2) and (-2, 4) respectively. The length of side AC is twice the length of side AB.

(i) Show that the complex number that represent vector $\overrightarrow{AB}$ is 4 + 2i. **1**

(ii) Find the complex number that the point C represents. **2**

(iii) Find the complex number that the point D represents such that ΔABD is an equilateral triangle. **2**

(e) Use the principle of mathematical induction to prove that: **3**

$$3^n > 1 + 2n \text{ where } n \text{ is an integer, } \quad n > 1$$

End of Question 12

Question 13 (15 marks) Start a new writing booklet

(a) Use the substitution $t = \tan\dfrac{x}{2}$ to find in simplest exact form of $\displaystyle\int \frac{1}{1+2\sin x-\cos x}\,dx$ **4**

(b) In the Argand plane, sketch $|z-1-3i| \le 2 \;\cap\; \dfrac{\pi}{4} < \arg z \le \dfrac{\pi}{2}$. **3**

(c) Find the primitive of $\dfrac{4x-1}{x^2+2x+6}$. **3**

(d) Find α and β given that $z^3+6z-4\sqrt{2}i=(z-\alpha)^2(z-\beta)$ **3**

(e) Consider the sphere S, centred at point $C(2, -1, 0)$ with radius $\sqrt{29}$. The line L with

parametric equations $\begin{cases} x=\lambda+1 \\ y=\lambda \\ z=2\lambda+3 \end{cases}$ intersects the surface of the sphere S at points P and Q.

(i) Find the coordinates of the points P and Q. **2**

(ii) A line parallel to line L is a tangent touching the sphere S at a single point, R. **2**

Find the possible coordinates of point R.

End of Question 13

Question 14 (15 marks) Start a new writing booklet

(a) Evaluate $\displaystyle\int_{\frac{1}{3}}^{1} \sqrt{x}\tan^{-1}\sqrt{x}\ dx.$ **3**

(b) Use the substitution of $x = 3\sin\theta$ to evaluate **4**

$$\int_{0}^{\frac{3}{\sqrt{2}}} \frac{dx}{\left(9-x^2\right)^{\frac{3}{2}}}$$

(c) A, B and C are three collinear points with position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ respectively. **2**

Point B lies between A and C with $\left|\overrightarrow{BC}\right| = \frac{1}{2}\left|\overrightarrow{AB}\right|$. Find $\underset{\sim}{c}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

(d)(i) Let x and y be real numbers such that $x \geq 0$ and $y \geq 0$. **1**

Prove that $\dfrac{x^2+y^2}{2} \geq xy$.

(ii) Suppose that a, b and c are real numbers, prove that $a^4+b^4+c^4 \geq a^2b^2+a^2c^2+b^2c^2$. **2**

(iii) Show that $a^2b^2+a^2c^2+b^2c^2 \geq a^2bc+b^2ac+c^2ab$. **2**

(iv) Deduce that if $a+b+c=d$, then $a^4+b^4+c^4 \geq abcd$. **1**

End of Question 14

Question 15 (15 marks) Start a new writing booklet

(a) (i) If $x \geq 0$, show that $\dfrac{x}{x^2+4} \leq \dfrac{1}{4}$. **2**

(ii) By integrating both sides of this inequality in part (i) with respect to x between the limits $x = 0$ and $x = \alpha$, show that **2**

$$e^{\frac{1}{2}\alpha} \geq \frac{1}{4}\alpha^2 + 1 \text{ for } \alpha \geq 0.$$

(b) Find the geometrical shape represented by the complex number z if $\omega = \dfrac{z-2}{z}$, **3**

given that ω is purely imaginary.

(c) Find $\displaystyle\int \sqrt{\frac{1-x}{1+x}}\, dx.$ **2**

(d) Let $z = e^{i\theta}$ where $z \neq 0$.

(i) Show that $z^n - \dfrac{1}{z^n} = 2i \sin n\theta$ for positive integers $n \geq 1$. **2**

(ii) Expand $\left(z - \dfrac{1}{z}\right)^5$ and show that $\sin^5\theta = \dfrac{1}{16}\left(\sin 5\theta - 5\sin 3\theta + 10\sin\theta\right)$. **3**

(iii) Find $\displaystyle\int \sin^5\theta\, d\theta.$ **1**

End of Question 15

Question 16 (15 marks) Start a new writing booklet

(a) If $P(x)=3x^4-11x^3+14x^2-11x+3$, show that **4**

$$P(x)=x^2\left\{3\left(x+\frac{1}{x}\right)^2-11\left(x+\frac{1}{x}\right)+8\right\}$$

and hence solve $P(x)=0$ over C and factorise $P(x)$ over R.

(b) Show that $\overline{z+w}=\overline{z}+\overline{w}$ for any complex numbers z and w. **2**

(c) Disprove this statement. **2**

"There exists $a\in N$ such that $a^2+9a+20$ is a prime number."

(d) Consider the lines L_1 and L_2 determined by vector equations

$$L_1:\ \underset{\sim}{r}=\begin{bmatrix}3\\2\\-1\end{bmatrix}+\lambda\begin{bmatrix}2\\-1\\1\end{bmatrix}\qquad\text{and}\qquad L_2:\ \underset{\sim}{r}=\begin{bmatrix}-1\\1\\0\end{bmatrix}+\mu\begin{bmatrix}1\\1\\-1\end{bmatrix}$$

(i) Show that L_1 and L_2 intersect and are perpendicular, stating the coordinates of the point of intersection. **3**

(ii) Deduce that the plane containing the lines L_1 and L_2 has an equation determined by **2**

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\3\\-2\end{bmatrix}+a\begin{bmatrix}2\\-1\\1\end{bmatrix}+b\begin{bmatrix}1\\1\\-1\end{bmatrix}$$ for parameters a and b, and hence that this plane has equation $y + z = 1$.

(iii) Find the perpendicular distance from the origin to this plane. **2**

End of Examination

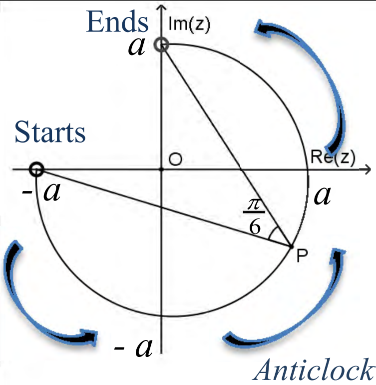
HGHS Ext 2 2022 TRIAL HSC SOLUTIONS

Marking Guidelines:

Multiple-choice Answer key

1	**A**
2	**B**
3	**C**
4	**D**
5	**D**
6	**C**
7	**A**
8	**B**
9	**C**
10	**C**

1	$\lvert \underset{\sim}{v} \rvert = \sqrt{5^2+(-3)^2+3^2}$ $= \sqrt{43}$ $\theta_i = \cos^{-1}\frac{5}{\sqrt{43}}$, $\theta_j = \cos^{-1}\left(\frac{-3}{\sqrt{43}}\right)$ and $\theta_k = \cos^{-1}\left(\frac{3}{\sqrt{43}}\right)$ $\therefore \theta_i = 40°19'$, $\theta_j = 117°14'$, and $\theta_k = 62°47'$ **(A)**	Correct answer: 83.33%
2.	$\frac{\left(1-\sqrt{3}\right)^6}{(1+i)^4} = \frac{\left(\sqrt{1+3}\right)^6 \text{cis } 6\left(-\frac{\pi}{3}\right)}{\left(\sqrt{1+1}\right)^4 \text{cis } 4\left(\frac{\pi}{4}\right)}$ $= \frac{2^6 \text{cis}(-2\pi)}{\left(\sqrt{2}\right)^4 \text{cis } \pi}$ $= \frac{64(1+0i)}{4(-1+0i)}$ $= -16$ **(B)**	Correct answer: 83.33%
3.	Statement: If P $\Rightarrow$ Q Converse: If Q $\Rightarrow$ P **(C)**	Correct answer: 88.89%
4.	$\int \frac{4x^3}{1+x^8}\,dx = \int \frac{4x^3}{1+\left(x^4\right)^2}\,dx$ $= \tan^{-1}\left(x^4\right) + C$ **(D)**	Correct answer: 88.89%

5.	 *Anticlockwise direction i.e. positive direction* So $\arg(z+ai)-\arg(z-a)=\dfrac{\pi}{6}$ $\arg\left(\dfrac{z+ai}{z-a}\right)=\dfrac{\pi}{6}$ **(D)**	Correct answer: 88.89%
6.	$\int \ln x \ dx = uv - \int u'v \ dx$ where $u=\ln x \quad v'=1$ $u=\dfrac{1}{x} \quad v'=x$ $=x\ln x-\int 1 \ dx$ $=x\ln x - x + C$ **(C)**	Correct answer: 100%
7.	$x^2+y^2+z^2-4x+10z+20=0$ $x^2-4x+y^2+z^2+10z=-20$ $x^2-4x+4+y^2+z^2+10z+25=-20+4+25$ $(x-2)^2+y^2+(z+5)^2=9$ Centre (2, 0, -5), radius = 3 units In vector form: $\left\lvert \underset{\sim}{r}-\begin{bmatrix}2\\0\\-5\end{bmatrix}\right\rvert=3$ **(A)**	Correct answer: 77.78%
8.	$e^{i3\theta}+e^{i\theta}=\text{cis}\,3\theta+\text{cis}\,\theta$ $=(\cos 3\theta+\cos\theta)+i(\sin 3\theta+\sin\theta)$ $=2\cos\left(\dfrac{3\theta+\theta}{2}\right)\cos\left(\dfrac{3\theta-\theta}{2}\right)$ $+2i\sin\left(\dfrac{3\theta+\theta}{2}\right)\cos\left(\dfrac{3\theta-\theta}{2}\right)$ $=2\cos 2\theta\cos\theta+2i\sin 2\theta\cos\theta$ $=2\cos\theta(\cos 2\theta+i\sin 2\theta)$ $=2\cos\theta\, e^{i2\theta}$ **(B)**	Correct answer: 61.11%

<table>
<tr>
<td>9.</td>
<td>.
<table>
<tr><th>Key words</th><th>Negation</th></tr>
<tr><td>$\exists x \in R$
$\ln x = 1$
and
$x > 2$</td><td>$\forall x \in R$
$\ln x \neq 1$
or
$x \leq 2.$</td></tr>
</table>
(C)</td>
<td>Correct answer: 88.89%</td>
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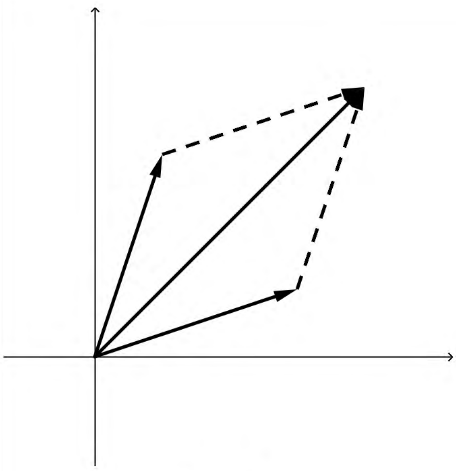
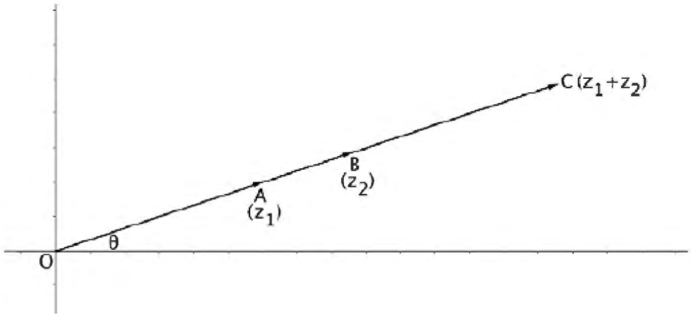
<td>10.</td>
<td>
$|\underset{\sim}{a} + \underset{\sim}{b}| \leq |\underset{\sim}{a}| + |\underset{\sim}{b}|$


Equality of that statement holds if and only if points O, A and B are collinear (C)</td>
<td>Correct answer: 38.89%</td>
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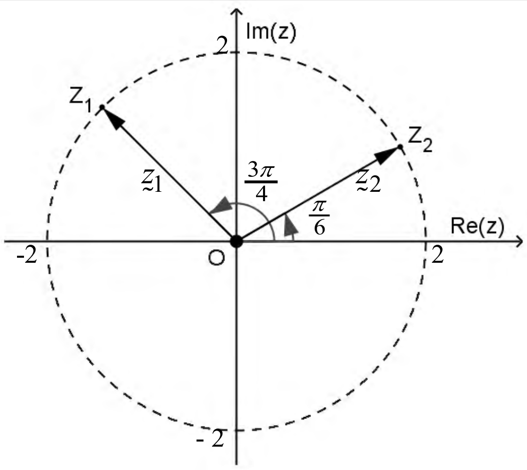
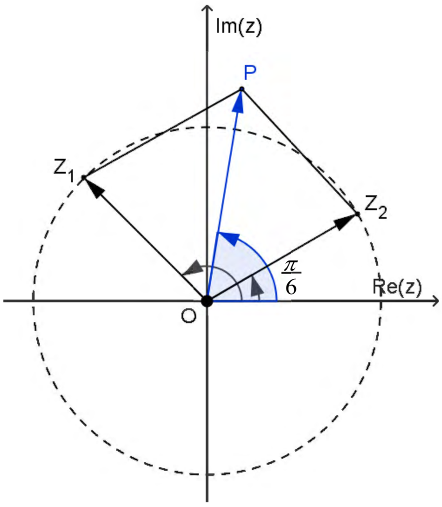
QUESTION 11 (a)

(i)	$. \ z_1 = -\sqrt{2} + \sqrt{2}i$ $\lvert z_1 \rvert = \sqrt{\left(-\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2}$ and $\arg z_1 = \tan^{-1}\left(\frac{\sqrt{2}}{-\sqrt{2}}\right)$ **(QII)** $= \sqrt{2+2}$ $\quad = \pi - \frac{\pi}{4}$ $= 2$ $\quad = \frac{3\pi}{4}$ $\therefore z_1 = 2\,\text{cis}\frac{3\pi}{4}$ $z_2 = \sqrt{3} + i$ $\lvert z_2 \rvert = \sqrt{\left(\sqrt{3}\right)^2 + \left(\sqrt{1}\right)^2}$ and $\arg z_2 = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ **(QI)** $= \sqrt{3+1}$ $\quad = \frac{\pi}{6}$ $= 2$ $\therefore z_2 = 2\,\text{cis}\frac{\pi}{6}$	**(QII) -** Need to pay attention to which component has a negative value as this affects the argument of the complex number.
(ii).		Even though the question specifically stated using vectors, only 44% complied. If using argument on the Argand diagram, students should have an open circle at the origin. This is poorly done. Future recommendation: since both moduli are of the same length, in future draw a circle in dashed line.
(iii)	$\therefore \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$ $= \frac{3\pi}{4} - \frac{\pi}{6}$ $= \frac{9\pi - 2\pi}{12}$ $= \frac{7\pi}{12}$	Mostly done well.

QUESTION 11 (a)

(iii)cont	$\angle Z_1OZ_2 = \dfrac{7\pi}{12}$ $\angle POZ_2 = \dfrac{7\pi}{24}$ $\therefore \arg(z_1 + z_2) = \dfrac{7\pi}{24} + \dfrac{\pi}{6} = \dfrac{11\pi}{24}$  	Please note: $\arg(z_1 + z_2) \neq \arg z_1 + \arg z_2$ Many students did not realise $\arg\left(\dfrac{z_1}{z_2}\right) = \angle Z_1OZ_2$. Hence they started from the Cartesian form and worked through leaving answers as $\tan^{-1}\left(\sqrt{3}+\sqrt{2}+\sqrt{6}+2\right)$ or unsimplified form $\tan^{-1}\left(\dfrac{\sqrt{2}+1}{\sqrt{3}-\sqrt{2}}\right)$ and in decimal value 1.439896…

QUESTION 11 (b)

	$\displaystyle\int \sin^3 x\cos^7 x\,dx = \int \sin^2 x\cos^7 x\sin x\,dx$ $\displaystyle= \int \left(1-\cos^2 x\right)\cos^7 x\sin x\,dx$ $\displaystyle= \int \left(\cos^7 x - \cos^9 x\right)\sin x\,dx$ $\displaystyle= -\int \left(\cos^7 x - \cos^9 x\right)(-\sin x)\,dx$ $\displaystyle= -\frac{\cos^8 x}{8} + \frac{\cos^{10} x}{10} + C$	Disappointing performance for a straight forward question.
	OR $\displaystyle\int \sin^3 x\cos^7 x\,dx = \int \sin^3 x\cos^3 x\cos^4 x\,dx$ $\displaystyle= \int (\sin x\cos x)^3\cos^4 x\,dx$ $\displaystyle= \frac{1}{8}\int (\sin 2x)^3\cos^4 x\,dx$ $\displaystyle= \frac{1}{8}\int \sin^3 2x\left(\cos^2 x\right)^2 dx$ $\displaystyle= \frac{1}{8}\int \sin^3 2x\left[\frac{1}{2}(1+\cos 2x)\right]^2 dx$	An attempt to make it a double angle trigonometry but got lost. – Not an efficient method.

QUESTION 11 (b) continues

	$$=\frac{1}{8}\int \sin^3 2x\left[\frac{1}{4}(1+\cos 2x)^2\right]dx$$ $$=\frac{1}{32}\int \sin^2 2x(1+\cos 2x)^2 \sin 2x\,dx$$ $$=\frac{1}{32}\int\left(1-\cos^2 2x\right)(1+\cos 2x)^2\sin 2x\,dx$$ $$=\frac{1}{32}\int\left(1-\cos^2 2x\right)\left(1+2\cos 2x+\cos^2 2x\right)\sin 2x\,dx$$ $$=\frac{1}{32}\int\left(1+2\cos 2x+\cos^2 2x-\cos^2 2x-2\cos^3 2x-\cos^4 2x\right)\sin 2x\,dx$$ $$=\frac{1}{32}\int\left(1+2\cos 2x-2\cos^3 2x-\cos^4 2x\right)\sin 2x\,dx$$ $$=\frac{1}{32}\int \sin 2x\,dx+\frac{1}{32}\int\left(2\cos 2x-2\cos^3 2x-\cos^4 2x\right)\sin 2x\,dx$$ $$=-\frac{1}{32}\left(\frac{\cos 2x}{2}\right)-\frac{1}{32}\int\left(2\cos 2x-2\cos^3 2x-\cos^4 2x\right)\sin 2x\,dx$$ $$=-\frac{\cos 2x}{64}-\frac{1}{32}\left(\frac{\sin^2 2x}{2\times 2}-2\frac{\left(\sin^4 2x\right)}{2\times 4}-\frac{\sin^5 2x}{2\times 5}\right)+C$$ $$=-\frac{\cos 2x}{64}-\frac{\sin^2 2x}{128}+\frac{\sin^4 2x}{128}+\frac{\cos^5 2x}{320}+C$$	

QUESTION 11 (c)

	$$\int\frac{dx}{\sqrt{e^{2x}-1}}\quad \text{where } u=\frac{1}{e^x}\quad \Leftrightarrow e^x=\frac{1}{u}$$ $$x=\ln\frac{1}{u}$$ $$x=-\ln u$$ $$\frac{dx}{du}=-\frac{1}{u}$$ $$dx=-\frac{1}{u}du$$ $$=\int\frac{-\frac{1}{u}du}{\sqrt{\left(\frac{1}{u}\right)^2-1}}$$	Few students got full mark. Area of concern are: (1) Students having substitution problem (2) Students not recognising that $\int -\frac{1}{\sqrt{1-u^2}}\,du=\cos^{-1}u+C$

QUESTION 11 (c)continues

	$= -\int \frac{\frac{1}{u}}{\sqrt{\frac{1}{u^2}-1}}\,du$ $= -\int \frac{\frac{1}{u}}{\sqrt{\frac{1-u^2}{u^2}}}\,du$ $= -\int \frac{\frac{1}{u}}{\frac{\sqrt{1-u^2}}{u}}\,du$ $= \int -\frac{1}{\sqrt{1-u^2}}\,du$ $= \cos^{-1} u + C$ $= \cos^{-1}\left(\frac{1}{e^x}\right) + C$	

QUESTION 11 (d)

(i)	$(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta$ $-10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$ By de Moirve's theorem, $\cos 5\theta + i\sin 5\theta = \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta$ $-10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$ Equating real terms, $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$ $= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$ $= \cos^5\theta - 10(\cos^3\theta - \cos^5\theta)$ $+5\cos\theta(1 - 2\cos^2\theta + \cos^4\theta)$ $= \cos^5\theta \quad -10\cos^3\theta$ $+10\cos^5\theta$ $+5\cos^5\theta - 10\cos^3\theta + 5\cos\theta$	Mostly done well.

QUESTION 11 (d) continues

<table>
<tr><td>(i)</td><td>$\therefore \cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ (as required)</td><td></td></tr>
<tr><td>(ii)</td><td>

$\cos 5\theta = -1$ for $0 \le \theta \le 2\pi$.

$5\theta = 2k\pi + \pi$, k = 0, 1, 2, 3, 4

$$\theta = \frac{2k\pi + \pi}{5} = \frac{(2k+1)\pi}{5}$$

When $k = 0$, $\theta = \frac{\pi}{5}$, one root is $\cos\frac{\pi}{5}$

When $k = 1$, $\theta = \frac{3\pi}{5}$, one root is $\cos\frac{3\pi}{5}$

When $k = 2$, $\theta = \frac{5\pi}{5} = \pi$, one root is $\cos\pi = -1$

When $k = 3$, $\theta = \frac{7\pi}{5}$, one root is $\cos\frac{7\pi}{5} = \cos\frac{3\pi}{5}$

When $k = 4$, $\theta = \frac{9\pi}{5}$, one root is $\cos\frac{9\pi}{5} = \cos\frac{\pi}{5}$

Given $\cos 5\theta = -1$

i.e. $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = -1$

$\therefore 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta + 1 = 0$

$a = 16$, $b = 0$, $c = -20$, $d = 0$, $e = 5$, $f = 1$

Sum of single roots:

$$\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} - 1 + \cos\frac{7\pi}{5} + \cos\frac{9\pi}{5} = -\frac{b}{a}$$

$$\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} - 1 + \cos\frac{3\pi}{5} + \cos\frac{\pi}{5} = 0$$

$$2\cos\frac{\pi}{5} + 2\cos\frac{3\pi}{5} = 1$$

$$2\left(\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}\right) = 1$$

$$\therefore \cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$$ (as required)

Product of 5 roots:

$$\left(\cos\frac{\pi}{5}\right)\left(\cos\frac{3\pi}{5}\right)(-1)\left(\cos\frac{7\pi}{5}\right)\left(\cos\frac{9\pi}{5}\right) = -\frac{f}{a}$$

</td><td>

Disappointing performance for a straight forward polynomial question.

Area of concern are:

(1) Students not relating equivalent trig. values

e.g. $\cos\frac{7\pi}{5} = \cos\frac{3\pi}{5}$

$\cos\frac{9\pi}{5} = \cos\frac{\pi}{5}$

(2) Students have forgotten sums and products of roots of polynomial.

</td></tr>
</table>

QUESTION 11 (d) continues

(ii)	$$-\left(\cos\frac{\pi}{5}\right)\left(\cos\frac{3\pi}{5}\right)\left(\cos\frac{3\pi}{5}\right)\left(\cos\frac{\pi}{5}\right)=-\frac{1}{16}$$ $$\left(\cos\frac{\pi}{5}\right)^2\left(\cos\frac{3\pi}{5}\right)^2=\frac{1}{16}$$ $$\left(\cos\frac{\pi}{5}\cos\frac{3\pi}{5}\right)^2=\frac{1}{16}$$ $$\cos\frac{\pi}{5}\cos\frac{3\pi}{5}=\pm\frac{1}{4}$$ Since $\cos\frac{\pi}{5}>0$ and $\cos\frac{3\pi}{5}<0$, $$\because \cos\frac{\pi}{5}\cos\frac{3\pi}{5}=-\frac{1}{4}$$	

QUESTION 12 (a)

	To prove by contradiction that $\log_5 7$ is irrational. <u>Proof</u>: Assume that $\log_5 7$ is rational i.e. $\exists\{p,\ q\}\in N,\ \log_5 7=\frac{p}{q}$ where p and q are relative primes. $q\log_5 7=p$ $\log_5 7^q=p$ $7^q=5^p$ LHS $=7^q$ $=7\times7\times7\times\ldots$ i.e. 7 is a factor of LHS RHS $=5^p$ $=5\times5\times5\times\ldots$ i.e. 5 is a factor of RHS but clearly neither 7 is not a factor of RHS nor 5 is a factor of LHS so there is a contradiction. Hence $\log_5 7$ is irrational.	Disappointing performance; Areas of concern: (1) students did not notice importance of the word :Assume" in the contradiction method. (2) Explanation needs to be succinct.

QUESTION 12 (b)

	Let $I=\int \sec^3 x \; dx$ $=\int \sec^2 x \sec x \; dx$ $=uv-\int u'v \, dx$ where $u=\sec x$ and $v'=\sec^2 x$ $u'=\sec x \tan x \quad v=\tan x$ $=\sec x \tan x-\int \sec x \tan^2 x \, dx$ $=\sec x \tan x-\int \sec x\left(\sec^2 x-1\right) dx$ $=\sec x \tan x-\int \left(\sec^3 x-\sec x\right) dx$ $=\sec x \tan x-\int \sec^3 x \, dx+\int \sec x \, dx$ $I=\sec x \tan x-I+\int \sec x \times \frac{\sec x+\tan x}{\sec x+\tan x} dx$ $2I=\sec x \tan x+\int \frac{\sec^2 x+\sec x \tan x}{\sec x+\tan x} dx$ $2I=\sec x \tan x+\ln\left\|\sec x+\tan x\right\|$ $\therefore I=\frac{1}{2}\sec x \tan x+\frac{1}{2}\ln\left\|\sec x+\tan x\right\|+C$	Disappointing performance Over 50% had problem with this.

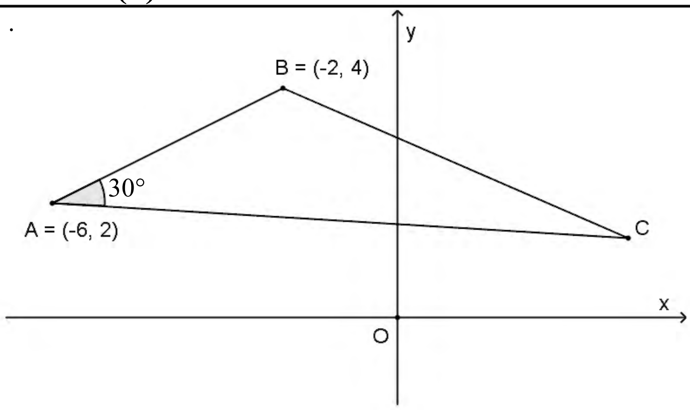
QUESTION 12 (c)

	Given ω is the complex cube root of unity i.e. $z^3=1$ $\omega^3=1$ $\omega^3-1=0$ $(\omega-1)\left(1+\omega+\omega^2\right)=0$ but $\omega \neq 1$ $\therefore 1+\omega+\omega^2=0$ $1+\omega=-\omega^2$ or $\omega+\omega^2=-1$ or $1+\omega^2=-\omega$	Mostly done well.

QUESTION 12 (c) continues

	$\therefore \left(1-3\omega+\omega^2\right)\left(1+\omega-8\omega^2\right)=\left(\left(1+\omega^2\right)-3\omega\right)\left(\left(1+\omega\right)-8\omega^2\right)$ $=\left(-\omega-3\omega\right)\left(-\omega^2-8\omega^2\right)$ $=\left(-4\omega\right)\left(-9\omega^2\right)$ $=36\omega^3$ and since $\omega^3=1$ $=36$	

QUESTION 12 (d)

(i)	 $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$ $=\begin{bmatrix}-2\\4\end{bmatrix}-\begin{bmatrix}-6\\2\end{bmatrix}$ $=\begin{bmatrix}4\\2\end{bmatrix}$ $\therefore \overrightarrow{AB}=4+2i$	Done well.
(ii)	$\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{AC}$ where $\left\|\overrightarrow{AC}\right\|=2\left\|\overrightarrow{AB}\right\|$ $\overrightarrow{AC}=2\,\text{cis}\left(-\frac{\pi}{6}\right)\left(4+2i\right)$ $=2\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)\left(4+2i\right)$ $=\left(\sqrt{3}-i\right)\left(4+2i\right)$ $=4\sqrt{3}+2\sqrt{3}i-4i+2$ $=\left(4\sqrt{3}+2\right)+\left(2\sqrt{3}-4\right)i$	Mostly done well. Areas of concern: (i) $\left\|\overrightarrow{AC}\right\|=2\left\|\overrightarrow{AB}\right\|$ does not mean $\overrightarrow{AC}=2\overrightarrow{AB}$ (ii) Some students are forgetting $\overrightarrow{AC}$ and $\overrightarrow{OC}$ are two different vectors. (iii) Some students did not pay attention to the fact that $\overrightarrow{AB}$ is rotated 30° clockwise hence $\left(-30^\circ\right)$ to obtain $\overrightarrow{AC}$.

QUESTION 12 (d) continues

<table>
<tr><td>(ii)cont.</td><td>

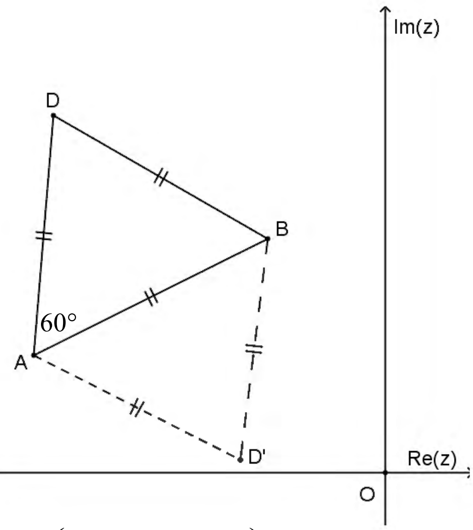
$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$

$\overrightarrow{OC} = (-6+2i) + \left[\left(4\sqrt{3}+2\right) + \left(2\sqrt{3}-4\right)i\right]$

$\therefore \overrightarrow{OC} = \left(4\sqrt{3}+2-6\right) + \left(2\sqrt{3}-4+2\right)i$

$\therefore \overrightarrow{OC} = \left(4\sqrt{3}-4\right) + \left(2\sqrt{3}-2\right)i$

</td><td>(iv) students should work in exact value rather than approximation.</td></tr>
<tr><td>(iii)</td><td>



$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$

$= \overrightarrow{OA} + \overrightarrow{AB}\,\mathrm{cis}\,\frac{\pi}{3}$

$= (-6+2i) + (4+2i)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= (-6+2i) + (4+2i)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$= (-6+2i) + (2+i)\left(1+\sqrt{3}i\right)$

$= (-6+2i) + 2 + 2\sqrt{3}i + i - \sqrt{3}$

$\therefore \overrightarrow{OD} = \left(-4-\sqrt{3}\right) + \left(2\sqrt{3}+3\right)i$

OR

$\overrightarrow{OD'} = \overrightarrow{OA} + \overrightarrow{AD'}$

$= \overrightarrow{OA} + \overrightarrow{AB}\,\mathrm{cis}\left(-\frac{\pi}{3}\right)$

$= (-6+2i) + (4+2i)\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

$= (-6+2i) + (4+2i)\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= (-6+2i) + (2+i)\left(1-\sqrt{3}i\right)$

$= (-6+2i) + 2 - 2\sqrt{3}i + i + \sqrt{3}$

$\therefore \overrightarrow{OD'} = \left(-4+\sqrt{3}\right) + \left(-2\sqrt{3}+3\right)i$

</td><td>Mostly done well.
Areas of concern:
(i) Assuming $\overrightarrow{AD} = \frac{1}{2}\overrightarrow{AC}$.
(ii) Some students are forgetting $\overrightarrow{AD}$ and $\overrightarrow{OD}$ are two different vectors.
(iii) Some students are not aware that there are two possible answers.</td></tr>
</table>

QUESTION 12 (e)

	Step 1: To prove true for $n = 2$ $\text{LHS} = 3^2$ $= 9$ $\text{RHS} = 2 = 2(2)+1$ $= 5$ $\therefore$ LHS > RHS Hence true for $n = 2$ Step 2: Assume true for $n = k$ where $k = 2, 3, 4, \ldots$ $3^k > 1 + 3k$ Step 3: To prove true for $n = k + 1$ $3^{k+1} > 1 + 3(k+1)$ $\text{LHS} = 3^{k+1}$ $= 3 \times 3^k$ $> 3(1+3k)$ Assumption for $n = k$ $> 3(1+k+2k)$ $> 3 \times 2k + 3(k+1)$ also $3 \times 2k > 1$ $> 1 + 3(k+1)$ $= \text{RHS}$ If true for $n = k$, hence proven true for $n = k + 1$. Step 4: Since true for $n = 2$, hence proven true for $n = 2 + 1 = 3$, $n = 3 + 1 = 4$, and so on. $\therefore$ true for all positive integers $n > 1$.	Mostly done well.

QUESTION 13 (a)

	$\displaystyle\int \frac{1}{1+2\sin x - \cos x}\,dx$ Let $t = \tan\dfrac{x}{2}$ $\dfrac{x}{2} = \tan^{-1} t$ $x = 2\tan^{-1} t$ $\dfrac{dx}{dt} = \dfrac{2}{1+t^2}$ $dx = \dfrac{2}{1+t^2}\,dt$ $\displaystyle = \int \frac{1}{1+2\left(\frac{2t}{1+t^2}\right) - \frac{1-t^2}{1+t^2}} \frac{2}{1+t^2}\,dt$ $\displaystyle = \int \frac{1}{1+\frac{4t}{1+t^2} - \frac{1-t^2}{1+t^2}} \frac{2}{1+t^2}\,dt$	Mostly done well. Areas of concern: (i) A few students did not recognise the fact that the denominator of integrand can be factorised, hence subsequent method to use is partial fraction. (ii) Some students did not learn how to differentiate $t = \tan\dfrac{x}{2}$ (iii) Some students did not even realised they have not completed because their answer is in terms of t and not x.

QUESTION 13 (a) continues

$$= \int \frac{1}{\dfrac{\left(1+t^2\right)+4t-\left(1-t^2\right)}{1+t^2}} \; \frac{2}{1+t^2}\,dt$$

$$= \int \frac{2}{1+t^2+4t-1+t^2} \; dt$$

$$= \int \frac{2}{2t^2+4t} \; dt$$

$$= \int \frac{1}{t^2+2t} \; dt$$

$$= \int \frac{1}{t\left(t+2\right)} \; dt$$

$$= \int \frac{A}{t}+\frac{B}{t+2}\,dt \quad \text{where } \frac{1}{t\left(t+2\right)} = \frac{A}{t}+\frac{B}{t+2}$$

$$1 = A\left(t+2\right)+Bt$$

when $t=0,\ 1=2A$

$$A=\frac{1}{2}$$

when $t=-2,\ 1=-2B$

$$B=-\frac{1}{2}$$

$$= \frac{1}{2}\int \frac{1}{t}-\frac{1}{t+2}\,dt$$

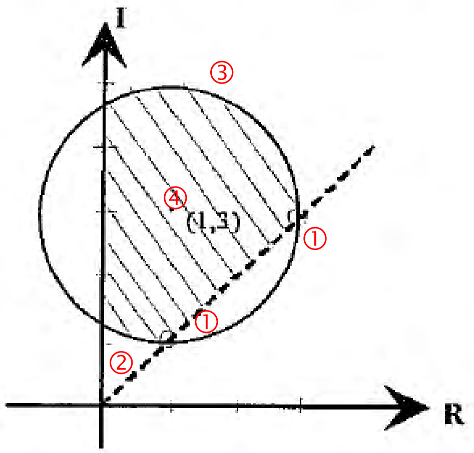
$$= \frac{1}{2}\left[\ln\left|t\right| - \ln\left|t+2\right|\right]+C$$

$$= \frac{1}{2}\left[\ln\left|\tan\frac{x}{2}\right| - \ln\left|\tan\frac{x}{2}+2\right|\right]+C$$

$$= \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| - \frac{1}{2}\ln\left|\tan\frac{x}{2}+2\right|+C \quad \text{or} \quad \frac{1}{2}\ln\left|\frac{\tan\frac{x}{2}}{\tan\frac{x}{2}+2}\right|+C$$

QUESTION 13 (b)

$|z-1-3i|\le 2 \;\cap\; \frac{\pi}{4}<\arg z\le\frac{\pi}{2}$



Mostly done well.
Key Areas of focus:
① open circles
② Region should not be shaded
③ Solid arch
④ Centre of circle (1, 3) and radius $\sqrt{2}$ units.

QUESTION 13 (c)

$$\int \frac{4x-1}{x^2+2x+6}\,dx = 2\int \frac{2x-\frac{1}{2}}{x^2+2x+6}\,dx$$

$$= 2\int \frac{2x+2-\frac{1}{2}-2}{x^2+2x+6}\,dx\,.$$

$$= 2\int \frac{(2x+2)-\frac{5}{2}}{x^2+2x+6}\,dx$$

$$= 2\int \frac{(2x+2)}{x^2+2x+6}\,dx - 2\int \frac{\frac{5}{2}}{x^2+2x+6}\,dx$$

$$= 2\ln\left|x^2+2x+6\right| - 2\left(\frac{5}{2}\right)\int \frac{1}{x^2+2x+6}\,dx$$

$$= 2\ln\left|x^2+2x+6\right| - 5\int \frac{1}{\left(x^2+2x+1\right)+6-1}\,dx$$

$$= 2\ln\left|x^2+2x+6\right| - 5\int \frac{1}{(x+1)^2+5}\,dx$$

where $f(x)=x+1$

$f'(x)=1$

$a=\sqrt{5}$

Mostly done well.

QUESTION 13 (c) continues

	$= 2\ln\left\|x^2+2x+6\right\| - \frac{5}{\sqrt{5}}\int \frac{1\times\sqrt{5}}{(x+1)^2+5}\,dx$ $= 2\ln\left\|x^2+2x+6\right\| - \sqrt{5}\tan^{-1}\frac{x+1}{\sqrt{5}} + C$	

QUESTION 13 (d)

	$z^3+6z-4\sqrt{2}i=(z-\alpha)^2(z-\beta)$ Let $P(z)=z^3+6z-4\sqrt{2}i$ i.e. $P(z)=(z-\alpha)^2(z-\beta)$ Since $P(z)$ has a double root, α. i.e. $P'(\alpha)=0$ $P'(z)=3z^2+6$ $3\alpha^2+6=0$ $3\alpha^2=-6$ $\alpha^2=-2$ $\therefore \alpha=\pm i\sqrt{2}$ $P(i\sqrt{2})=(i\sqrt{2})^3+6(i\sqrt{2})-4\sqrt{2}i$ $=-2\sqrt{2}i+6\sqrt{2}i-4\sqrt{2}i$ $\therefore P(i\sqrt{2})=0$ $P(-i\sqrt{2})=(-i\sqrt{2})^3+6(-i\sqrt{2})-4\sqrt{2}i$ $=2\sqrt{2}i-6\sqrt{2}i-4\sqrt{2}i$ $\therefore P(-i\sqrt{2})\neq 0$ $\therefore i\sqrt{2}$ is a root. Let $P(z)=z^3+6z-4\sqrt{2}i=0$ a=1 b-0, c=6, d= $-4\sqrt{2}i$ $\alpha+\alpha+\beta=-\frac{b}{a}$ $2\alpha+\beta=0$ $\therefore \beta=-2\alpha$ $\therefore \beta=-2\sqrt{2}i$	Mostly done well. Area of concern: Many students did not check that only $i\sqrt{2}$ **is** the only root. **They** assumed both values of α are valid roots of $P(z)$.

QUESTION 13 (e)

(i)	$x = \lambda + 1$ $y = \lambda$ $z = 2\lambda + 3$ $\lvert v - c \rvert = \sqrt{29}$ $\left\lvert \begin{pmatrix} \lambda+1 \\ \lambda \\ 2\lambda+3 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\rvert = \sqrt{29}$ $\left\lvert \begin{pmatrix} \lambda-1 \\ \lambda+1 \\ 2\lambda+3 \end{pmatrix} \right\rvert = \sqrt{29}$ $(\lambda-1)^2 + (\lambda+1)^2 + (2\lambda+3)^2 = 29$ $6\lambda^2 + 12\lambda - 18 = 0$ $\lambda^2 + 2\lambda - 3 = 0$ $\lambda = -3,\ \lambda = 1$ $\therefore P: (-2, -3, -3),\ Q: (2, 1, 5)$	A few students did not know what to do.
(ii)	Since the line is parallel to L and tangent to the sphere at R, CR will be perpendicular to L and bisect PQ. Let M be the mid-point of PQ and, hence, the intersection point of PQ and CR. Therefore: $P: (-2, -3, -3),\ Q: (2, 1, 5),\ C: (2, -1, 0)$ $M: (0, -1, 1)$ $\overrightarrow{CM} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ As $\overrightarrow{CR}$ is an extension of $\overrightarrow{CM}$ with a length of $\sqrt{29}$: $\overrightarrow{CR} = \sqrt{\frac{29}{5}} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ $\overrightarrow{OR} = \sqrt{\frac{29}{5}} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ $R: \left(2 - 2\sqrt{\frac{29}{5}},\ -1,\ \sqrt{\frac{29}{5}} \right)$	Poor performance.

QUESTION 14 (a)

	$\displaystyle\int_{\frac{1}{3}}^{1} \sqrt{x}\tan^{-1}\sqrt{x}\ dx$ $\displaystyle=\left[uv\right]_{\frac{1}{3}}^{1}-\int_{\frac{1}{3}}^{1} u'v\ dx \quad \text{where } u=\tan^{-1}\left(x^{\frac{1}{2}}\right) \text{ and } v'=x^{\frac{1}{2}}$ $\displaystyle u'=\frac{\frac{1}{2}x^{-\frac{1}{2}}}{1+\left(\sqrt{x}\right)^2} \qquad v=\frac{2x^{\frac{3}{2}}}{3}$ $\displaystyle u'=\frac{\frac{1}{2\sqrt{x}}}{1+x}$ $\displaystyle u'=\frac{1}{2\sqrt{x}\left(1+x\right)}$ $\displaystyle=\frac{2}{3}\left[x\sqrt{x}\tan^{-1}\sqrt{x}\right]_{\frac{1}{3}}^{1}-\frac{2}{3}\int_{\frac{1}{3}}^{1}\frac{1}{2\sqrt{x}\left(1+x\right)}\left(x\sqrt{x}\right)\ dx$ $\displaystyle=\frac{2}{3}\left[(1)\sqrt{(1)}\tan^{-1}\sqrt{(1)}-\left(\frac{1}{3}\right)\sqrt{\left(\left(\frac{1}{3}\right)\right)}\tan^{-1}\sqrt{\left(\left(\frac{1}{3}\right)\right)}\right]-\frac{1}{3}\int_{\frac{1}{3}}^{1}\frac{x}{1+x}\ dx$ $\displaystyle=\frac{2}{3}\left[\tan^{-1}1-\frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)\tan^{-1}\frac{1}{\sqrt{3}}\right]-\frac{1}{3}\int_{\frac{1}{3}}^{1}\frac{(x+1)-1}{1+x}\ dx$ $\displaystyle=\frac{2}{3}\left[\frac{\pi}{4}-\frac{1}{3\sqrt{3}}\left(\frac{\pi}{6}\right)\right]-\frac{1}{3}\int_{\frac{1}{3}}^{1}1-\frac{1}{1+x}\ dx$ $\displaystyle=\frac{\pi}{6}-\frac{\pi}{27\sqrt{3}}-\frac{1}{3}\left[x-\ln\lvert 1+x\rvert\right]_{\frac{1}{3}}^{1}$ $\displaystyle=\frac{\pi}{6}-\frac{\pi}{27\sqrt{3}}-\frac{1}{3}\left[(1)-\ln\lvert 1+(1)\rvert-\left(\left(\frac{1}{3}\right)-\ln\left\lvert 1+\left(\frac{1}{3}\right)\right\rvert\right)\right]$ $\displaystyle=\frac{\pi}{6}-\frac{\pi}{27\sqrt{3}}-\frac{1}{3}\left[1-\ln 2-\left(\frac{1}{3}-\ln\frac{4}{3}\right)\right]$ $\displaystyle=\frac{\pi}{6}-\frac{\pi}{27\sqrt{3}}-\frac{1}{3}\left[\frac{2}{3}+\ln\left(2\left(\frac{4}{3}\right)\right)\right]$ $\displaystyle=\frac{\pi}{6}-\frac{\pi}{27\sqrt{3}}-\frac{1}{3}\left[\frac{2}{3}+\ln\left(\frac{8}{3}\right)\right]$	Poor performance due to careless arithmetic mistakes and transcription error. A few students still need to learn to recognise when to use integration by parts.

QUESTION 14 (a)cont.

	$=\frac{\pi}{6}-\frac{\pi}{27\sqrt{3}}-\frac{2}{9}-\frac{1}{3}\ln\left(\frac{8}{3}\right)$	

QUESTION 14 (b)

	$\int_0^{\frac{3}{\sqrt{2}}}\frac{dx}{\left(9-x^2\right)^{\frac{3}{2}}}$ where $x=3\sin\theta$ $\frac{dx}{d\theta}=3\cos\theta$ $dx=3\cos\theta\,d\theta$ When $x=\frac{3}{\sqrt{2}}$, $\theta=\frac{\pi}{4}$ $x=0 \quad \theta=0$ $=3\int_0^{\frac{\pi}{4}}\frac{\cos\theta\,d\theta}{\left(9-9\sin^2\theta\right)^{\frac{3}{2}}}$ $=3\int_0^{\frac{\pi}{4}}\frac{\cos\theta\,d\theta}{\left(9\left(1-\sin^2\theta\right)\right)^{\frac{3}{2}}}$ $=3\int_0^{\frac{\pi}{4}}\frac{\cos\theta\,d\theta}{\left(3^2\cos^2\theta\right)^{\frac{3}{2}}}$ $=3\int_0^{\frac{\pi}{4}}\frac{\cos\theta}{27\cos^3\theta}\,d\theta$ $=\frac{1}{9}\int_0^{\frac{\pi}{4}}\frac{1}{\cos^2\theta}\,d\theta$ $=\frac{1}{9}\int_0^{\frac{\pi}{4}}\sec^2\theta\,d\theta$ $=\frac{1}{9}\left[\tan\theta\right]_0^{\frac{\pi}{4}}$	Mostly done well. Area of concern: $\frac{1}{9\cos^2 x}=9\sec^2 x$

QUESTION 14 (b) continues

	$= \frac{1}{9}\left[\tan\frac{\pi}{4} - 0\right]$ $= \frac{1}{9}$	

QUESTION 14 (c)

	$\lvert\overrightarrow{BC}\rvert = \frac{1}{2}\lvert\overrightarrow{AB}\rvert$ (B divides AC into a ratio of $2:1$) $\underset{\sim}{c} - \underset{\sim}{b} = \frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{a})$ $\underset{\sim}{c} = \frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{a}) + \underset{\sim}{b}$ $= \frac{3}{2}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}$	Mostly done well.

QUESTION 14 (d)

(i)	For all $x \geq 0$, $y \geq 0$, $(x-y)^2 \geq 0$ $x^2 - 2xy + y^2 \geq 0$ $x^2 + y^2 \geq 2xy$ $\therefore \frac{x^2+y^2}{2} \geq xy$ (as required)	Mostly done well.
(ii)	Substituting $x = a^2$ and $y = b^2$ into part (i), $\frac{(a^2)^2 + (b^2)^2}{2} \geq (a^2)(b^2)$ $\frac{a^4+b^4}{2} \geq a^2b^2$ $\therefore a^4 + b^4 \geq 2a^2b^2$ — ① Similarly $b^4 + c^4 \geq 2b^2c^2$ — ② $c^4 + a^4 \geq 2c^2a^2$ — ③ ① + ② + ③ $2a^4 + 2b^4 + 2c^4 \geq 2a^2b^2 + 2b^2c^2 + 2c^2a^2$ $\therefore a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2$ (as req'd)	Mostly done well.

QUESTION 14 (d)

(iii)	From part (i) $x^2+y^2 \ge 2xy$ Let $x=a$ and $y=b$, $a^2+b^2 \ge 2ab$ $c^2\left(a^2+b^2\right) \ge 2abc^2$ — ① Similarly $a^2\left(b^2+c^2\right) \ge 2bca^2$ — ② $b^2\left(c^2+a^2\right) \ge 2cab^2$ — ③ ① + ② + ③ $c^2\left(a^2+b^2\right)+a^2\left(b^2+c^2\right)+b^2\left(c^2+a^2\right) \ge 2\left(abc^2+bca^2+cab^2\right)$ $c^2a^2+c^2b^2+a^2b^2+a^2c^2+b^2c^2+b^2a^2 \ge 2\left(abc^2+bca^2+cab^2\right)$ $2a^2b^2+2b^2c^2+2c^2a^2 \ge 2\left(abc^2+bca^2+cab^2\right)$ $\therefore a^2b^2+b^2c^2+c^2a^2 \ge c^2ab+a^2bc+b^2ca$ (as required)	Students that know it, permed well. 40% did not know what to do.
(iv)	From part (ii) $a^4+b^4+c^4 \ge a^2b^2+b^2c^2+c^2a^2$ From part (iii) $a^2b^2+b^2c^2+c^2a^2 \ge c^2ab+a^2bc+b^2ca$ $a^4+b^4+c^4 \ge a^2b^2+b^2c^2+c^2a^2 \ge c^2ab+a^2bc+b^2ca$ $\therefore a^4+b^4+c^4 \ge c^2ab+a^2bc+b^2ca$ $\therefore a^4+b^4+c^4 \ge cab\left(c+a+b\right)$ Since $a+b+c=d$ $\therefore a^4+b^4+c^4 \ge cabd$ (as required)	Students that know it, permed well. 40% did not know what to do.

QUESTION 15 (a)

(i)	$\left(x-2\right)^2 \ge 0, \quad x \ge 0$ $x^2-4x+4 \ge 0$ $x^2+4 \ge 4x$ $1 \ge \dfrac{4x}{x^2+4}$ $\dfrac{1}{4} \ge \dfrac{x}{x^2+4}$ $\therefore \dfrac{x}{x^2+4} \le \dfrac{1}{4}$ (as required)	Mostly done well. Area of concern: students not mentioning $x \ge 0$

QUESTION 15 (a)

(ii)	$\dfrac{x}{x^2+4} \le \dfrac{1}{4}$ [from part (i)] $\displaystyle\int_0^\alpha \frac{x}{x^2+4}\,dx \le \int_0^\alpha \frac{1}{4}\,dx$ where $\alpha \ge 0$ $\displaystyle\frac{1}{2}\int_0^\alpha \frac{2x}{x^2+4}\,dx \le \frac{1}{4}\int_0^\alpha 1\,dx$ $\dfrac{1}{2}\left[\ln\left(x^2+4\right)\right]_0^\alpha \le \dfrac{1}{4}[x]_0^\alpha$ $\ln\left(\alpha^2+4\right)-\ln 4 \le \dfrac{1}{2}\alpha$ $\ln\left(\dfrac{\alpha^2+4}{4}\right) \le \dfrac{1}{2}\alpha$ $\dfrac{\alpha^2+4}{4} \le e^{\frac{1}{2}\alpha}$ (Since $f(x)=e^x$ is an increasing function) $\dfrac{\alpha^2}{4}+\dfrac{4}{4} \le e^{\frac{1}{2}\alpha}$ $\therefore e^{\frac{1}{2}\alpha} \ge \dfrac{\alpha^2}{4}+1$ (as required)	Mostly done well. Area of concern: Poor technique with log. Law. $\frac{1}{2}\ln\left(x^2+4\right) \ne \frac{1}{2}\ln x^2 + \ln 2$

QUESTION 15 (b)

	$\omega = \dfrac{z-2}{z}$ Let $z = x+iy$ $\omega = 1-\dfrac{2}{z}$ $=1-\dfrac{2}{x+iy}\times\dfrac{x-iy}{x-iy}$ $=1-\dfrac{2(x-iy)}{x^2+y^2}$ $=1-\dfrac{2x}{x^2+y^2}+\dfrac{2iy}{x^2+y^2}$ Since ω is purely imaginary, i.e. $\text{Re}(\omega)=0$ $\therefore 1-\dfrac{2x}{x^2+y^2}=0$	Many students did not know what to do.

QUESTION 15 (b) continues

	$$\frac{x^2+y^2-2x}{x^2+y^2}=0$$ $$x^2+y^2-2x=0$$ $$x^2-2x+y^2=0$$ $$x^2-2x+1+y^2=0+1$$ $$(x-1)^2+y^2=1$$ $$\therefore \lvert z-1\rvert=1$$ Hence the locus of z is a circle of radius 1 and centre (1, 0)	

QUESTION 15 (c)

	$$\int\sqrt{\frac{1-x}{1+x}}\,dx=\int\sqrt{\frac{1-x}{1+x}\times\frac{1-x}{1-x}}\,dx$$ $$=\int\sqrt{\frac{(1-x)^2}{1-x^2}}\,dx$$ $$=\int\frac{1-x}{\sqrt{1-x^2}}\,dx$$ $$=\int\frac{1}{\sqrt{1-x^2}}-\frac{x}{\sqrt{1-x^2}}\,dx$$ $$=\int\frac{1}{\sqrt{1-x^2}}-x\left(1-x^2\right)^{-\frac{1}{2}}\,dx$$ $$=\int\frac{1}{\sqrt{1-x^2}}\,dx-\int x\left(1-x^2\right)^{-\frac{1}{2}}\,dx$$ $$=\sin^{-1}x+\frac{1}{2}\int -2x\left(1-x^2\right)^{-\frac{1}{2}}\,dx$$ $$=\sin^{-1}x+\left(1-x^2\right)^{\frac{1}{2}}+C\ \text{ or }\ =\sin^{-1}x+\sqrt{1-x^2}+C$$	

QUESTION 15 (d)

(i)	$z^n = (\cos\theta + i\sin\theta)^n$ $= \cos n\theta + i\sin n\theta$ (de Moirve's theorem) $\dfrac{1}{z^n} = (\cos\theta + i\sin\theta)^{-n}$ $= \cos(-n)\theta + i\sin(-n)\theta$ $= \cos n\theta - i\sin n\theta$ $\therefore z^n - \dfrac{1}{z^n} = (\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta)$ $= \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta$ $\therefore z^n - \dfrac{1}{z^n} = 2i\sin n\theta$ (as required)	
(ii)	$\left(z-\dfrac{1}{z}\right)^5 = \binom{5}{0}z^5 - \binom{5}{1}z^4\left(\dfrac{1}{z}\right) + \binom{5}{2}z^3\left(\dfrac{1}{z}\right)^2 + \binom{5}{3}z^2\left(\dfrac{1}{z}\right)^3 + \binom{5}{4}z\left(\dfrac{1}{z}\right)^4 - \binom{5}{5}\left(\dfrac{1}{z}\right)^5$ $= z^5 - 5z^3 + 10z - 10\left(\dfrac{1}{z}\right) + 5\left(\dfrac{1}{z}\right)^3 - \left(\dfrac{1}{z}\right)^5$ $\left(z-\dfrac{1}{z}\right)^5 = \left(z^5 - \dfrac{1}{z^5}\right) - 5\left(z^3 - \dfrac{1}{z^3}\right) + 10\left(z - \dfrac{1}{z}\right)$ $(2i\sin\theta)^5 = (2i\sin 5\theta) - 5(2i\sin 3\theta) + 10(2i\sin\theta)$ $32i\sin^5\theta = 2i(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$ $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$ $\therefore \sin^5\theta = \dfrac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$ (as required)	
(iii)	$\therefore \int \sin^5\theta\, d\theta = \dfrac{1}{16}\int (\sin 5\theta - 5\sin 3\theta + 10\sin\theta)\, d\theta$ $= \dfrac{1}{16}\left(-\dfrac{\cos 5\theta}{5} + \dfrac{5\cos 3\theta}{3} - 10\cos\theta\right) + C$ $= -\dfrac{\cos 5\theta}{80} + \dfrac{5\cos 3\theta}{48} - \dfrac{5\cos\theta}{8} + C$	

QUESTION 16 (a)

$$P(x) = 3x^4 - 11x^3 + 14x^2 - 11x + 3$$

$$= x^2\left(3x^2 - 11x + 14 - \frac{11}{x} + \frac{3}{x^2}\right)$$

$$= x^2\left\{\left(3x^2 + \frac{3}{x^2}\right) - \left(11x + \frac{11}{x}\right) + 14\right\}$$

$$= x^2\left\{3\left(x^2 + \frac{1}{x^2}\right) - 11\left(x + \frac{1}{x}\right) + 14\right\}$$

Since $\left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2}$

$$= x^2 + \frac{1}{x^2} + 2$$

$$x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$$

$$\therefore P(x) = x^2\left\{3\left(\left(x + \frac{1}{x}\right)^2 - 2\right) - 11\left(x + \frac{1}{x}\right) + 14\right\}$$

$$= x^2\left\{3\left(x + \frac{1}{x}\right)^2 - 6 - 11\left(x + \frac{1}{x}\right) + 14\right\}$$

$$\therefore P(x) = x^2\left\{3\left(x + \frac{1}{x}\right)^2 - 11\left(x + \frac{1}{x}\right) + 8\right\} \text{ (as required)}$$

For $P(x) = 0$

$$x^2\left\{3\left(x + \frac{1}{x}\right)^2 - 11\left(x + \frac{1}{x}\right) + 8\right\} = 0$$

$$x\left[3\left(x + \frac{1}{x}\right) - 8\right] \times x\left[\left(x + \frac{1}{x}\right) - 1\right] = 0$$

$$x\left[3\left(\frac{x^2+1}{x}\right) - 8\right] = 0 \quad \text{or} \quad x\left[\left(\frac{x^2+1}{x}\right) - 1\right] = 0$$

$$3\left(x^2 + 1\right) - 8x = 0 \qquad x^2 - x + 1 = 0$$

$$3x^2 - 8x + 3 = 0 \qquad x = \frac{1 \pm \sqrt{1-4}}{2}$$

$$x = \frac{8 \pm \sqrt{64 - 4(3)(3)}}{2(3)} \qquad x = \frac{1 \pm \sqrt{-3}}{2}$$

$$x = \frac{8 \pm 2\sqrt{7}}{6} \qquad x = \frac{1 \pm i\sqrt{3}}{2}$$

QUESTION 16 (a) continues

	$x=\dfrac{4\pm\sqrt{7}}{3}$ $\therefore x=\dfrac{4\pm\sqrt{7}}{3}$ or $\dfrac{1\pm i\sqrt{3}}{2}$ $\therefore P(x)=\left(x-\dfrac{4-\sqrt{7}}{3}\right)\left(x-\dfrac{4+\sqrt{7}}{3}\right)\left(x^2-x+1\right)$ over R	

QUESTION 16 (b)

	Let $z=a+ib$ $w=c+id$ where a, b, c and d are real. $\overline{z+w}=\overline{(a+ib)+(c+id)}$ $=\overline{(a+c)+(b+d)i}$ $=(a+c)-(b+d)i$ $=(a-ib)+(c-id)$ $=\bar{z}+\bar{w}$	

QUESTION 16 (c)

	Required to prove that, for all $a\in\mathbb{N}$ such that $a^2+9p+20$ is a prime number. Now $a^2+9a+20=(a+4)(a+5)$ Clearly this is a composite number. $\therefore a^2+9p+20$ is *not* a prime number.	

QUESTION 16 (d)

(i)	$3+2\lambda=-1+\mu$ — ① $2-\lambda=1+\mu$ — ② $-1+\lambda=-\mu$ — ③ ①-② $1+3\lambda=-2$ $3\lambda=-3$ $\therefore \lambda=-1$ — ④ Sub ④ into ① $3-2=-1+\mu$ $\therefore \mu=2$ Sub $\lambda=-1$ and $\mu=2$ into ③, LHS = -2	

QUESTION 16 (d) continues

<table>
<tr><td>(i)</td><td>

RHS = -2

$\therefore$ LHS = RHS

Hence there are a pair of unique values of λ and μ that satisfy the three equations simultaneously, hence the lines intersects at a common point.

At intersection, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$

Also $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 2-1-1=0$

Hence the lines intersect at right angles at (1, 3, -2).

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<tr><td>(ii)</td><td>

Unit vectors in the directions of the perpendicular vectors $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ from the basis of a plane in 3D space so that linear combinations of such vectors determine a plane through the origin. Addition of the vector $\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ translate the plane to pass through the point of intersection of the lines L_1 and L_2.

Hence the unique plane containing L_1 and L_2 has equation

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + a\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ for parameters a and b.

Note that b = 0 gives a vector equation L_1 and a = 0 gives a vector equation for L_2 since (1, 3, -2) lies on both lines.

For all points on this plane, $\begin{aligned} x &= 1+2a+b \ldots(1) \\ y &= 3-a+b \ldots \quad (2) \\ z &= -2+a-b \ldots \quad (3) \end{aligned}$

(2) + (3) gives $y+z=1$

Hence the equation of the plane is $y+z=1$.

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QUESTION 16 (d)

(iii)	**Method 1:** Let $P(x, y, 1 - y)$ be a point in the plane $y + z = 1$. The perpendicular distance from the origin to this plane is the square root of the minimum value of $\|OP\|^2 = x^2 + y^2 + (1 - y)^2$ $\|OP\|^2 = x^2 + 2\left(y - \frac{1}{2}\right)^2 + \frac{1}{2}$. The expression has a minimum value of $\frac{1}{2}$ when $x = 0$ and $y = \frac{1}{2}$. Hence the perpendicular distance from the origin to the plane $y + z = 1$ is $\frac{1}{\sqrt{2}}$ units. **Method 2:** Consider the vector $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$ perpendicular to both $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. $2u - v + w = 0$ ...(1) $u + v - w = 0$ ... (2) (1) + (2) $3u = 0$ $\therefore u = 0$ $\therefore v = w$ Hence $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is a vector through the origin which is perpendicular to both $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, and hence to the plane defined in (ii), and meets this plane at the point where $x = 0$ and $y = z$ and $y + z = 1$, that is the point $\left(0, \frac{1}{2}, \frac{1}{2}\right)$. Hence the perpendicular distance from the origin to the plane in (ii) is $\sqrt{0 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$ units. **THE END**	